

General relativity histories theory II: Invariance groups.

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February 6, 2008

Abstract

In this paper we show in detail how the histories description of general relativity carries representations of both the spacetime diffeomorphisms group and the Dirac algebra of constraints. We show that the introduction of metric-dependent equivariant foliations leads to the crucial result that the canonical constraints are invariant under the action of spacetime diffeomorphisms. Furthermore, there exists a representation of the group of generalised spacetime mappings that are functionals of the four-metric: this is a spacetime analogue of the group originally defined by Bergmann and Komar in the context of the canonical formulation of general relativity. Finally, we discuss the possible directions for the quantization of gravity in histories theory.

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1 Introduction

The work presented here is a continuation of [1] in which we discussed the covariant and the canonical description of histories general relativity. A key ingredient of [1] was the introduction of metric-dependent foliations in order to preserve the spacelike character of a foliation with respect to a Lorentzian four-metric g .

The aim of the present paper is to address the second major issue of the canonical formalism: namely, the degree to which physical results depend upon the choice of a Lorentzian foliation. For each choice of foliation, solutions to the canonical equations of motion yield different 4-metrics. If different such descriptions are to be equivalent, the corresponding 4-metrics should be related by spacetime diffeomorphisms. We must show, therefore, that the action of the spacetime diffeomorphisms group intertwines between constructions corresponding to different choices of the foliation.

To this end, we will discuss the relation between the two major invariance groups of gravity, namely, the group of spacetime diffeomorphisms $\text{Diff}(M)$, and the (canonical) Dirac algebra of constraints. The key result is that the natural requirement of the equivalence between descriptions corresponding to different choices of foliation, can be expressed by a simple mathematical condition, which we shall call the *equivariance condition*¹.

In particular, given a tensor field $A(\cdot, g]$ that is parameterised by a Lorentzian metric g , the possibility arises that, under a diffeomorphism transformation, the usual induced transformation of A can be compensated by the additional change arising from the functional dependence on g . Specifically, we say that the tensor function $g \mapsto A(g)$ is *equivariant* if

$$f^*A(\cdot, g] = A(\cdot, f^*g] \quad (1.1)$$

for all Lorentzian metrics g and $f \in \text{Diff}(M)$; here, f^* denotes the usual pull-back operation on tensor fields.

Of particular importance in what follows is the analogous notion of an *equivariant foliation*. Specifically, we say that a metric-dependent foliation

¹The term ‘equivariance’ is usually employed in the following situation. If two spaces M and N carry actions α and β respectively of a group G , then a map $f : M \rightarrow N$ is *equivariant* with respect to these actions, if $\beta(g)f(x) = f(\alpha(g)x)$, for all $x \in M$, $g \in G$. In the present context the group G is $\text{Diff}(M)$.

$\mathcal{E}(g) : \mathbb{R} \times \Sigma \rightarrow M$ is equivariant if

$$\mathcal{E}(f^*g) = f^{-1} \circ \mathcal{E}(g) \quad (1.2)$$

for all Lorentzian metrics g and $f \in \text{Diff}(M)$.

The introduction of equivariant metric-dependent embeddings leads to a very significant result: the Hamiltonian constraints, the canonical action functional, and the equations of motion on the reduced state space are all invariant under the action of the group of spacetime diffeomorphisms.

In addition to the usual canonical and spacetime covariance groups, Bergmann and Komar showed that one may define a group of generalised spacetime diffeomorphisms that have a functional dependence on the four-metric g [2]; in what follows we shall denote this by $\mathcal{BK}(M)$. We shall show that a representation of this group also exists in histories theory, and we will discuss its relation with the other two groups.

1.1 Relation between the spacetime and the canonical general relativity description

The history space for general relativity is defined as $\Pi^{cov} = T^*\text{LRiem}(M)$, where $\text{LRiem}(M)$ is the space of all Lorentzian four-metrics $g_{\mu\nu}$, on a four-dimensional manifold² M , and $T^*\text{LRiem}(M)$ is its cotangent bundle. It is equipped with the symplectic form $\Omega = \int d^4X \delta\pi^{\mu\nu} \wedge \delta g_{\mu\nu}$ where $X \in M$, and $g_{\mu\nu}(X) \in \text{LRiem}(M)$, and $\pi^{\mu\nu}(X)$ is the conjugate variable.

The symplectic structure Eq. (1.9) generates the covariant Poisson brackets algebra,

$$\{g_{\mu\nu}(X), g_{\alpha\beta}(X')\} = 0 \quad (1.3)$$

$$\{\pi^{\mu\nu}(X), \pi^{\alpha\beta}(X')\} = 0 \quad (1.4)$$

$$\{g_{\mu\nu}(X), \pi^{\alpha\beta}(X')\} = \delta_{(\mu\nu)}^{\alpha\beta} \delta^4(X, X'), \quad (1.5)$$

where we have defined $\delta_{(\mu\nu)}^{\alpha\beta} := \frac{1}{2}(\delta_\mu^\alpha \delta_\nu^\beta + \delta_\mu^\beta \delta_\nu^\alpha)$.

For a fixed metric g we can choose a foliation to be spacelike, in the sense that $t \mapsto h_{ij}(t, \underline{x})$ is a path in the space of Riemannian metrics on Σ . However, this foliation will fail to be spacelike for certain other Lorentzian metrics on

²We will assume that M has the topology of $\mathbb{R} \times \Sigma$ for some three-manifold Σ .

M . This is not important at the level of the classical theory, because we generally only consider four-metrics are solutions to the equations of motion; however it is a non-trivial issue in the quantum theory.

In order to address this issue we introduce the space of metric-dependent foliations. For a given Lorentzian metric g , we use the foliation $\mathcal{E}(g)$ to split g with respect to the Riemannian three-metric h_{ij} , the lapse function N and the shift vector N^i as

$$h_{ij}(t, \underline{x}) := \mathcal{E}_{,i}^\mu(t, \underline{x}; g) \mathcal{E}_{,j}^\nu(t, \underline{x}; g) g_{\mu\nu}(\mathcal{E}(t, \underline{x}; g)) \quad (1.6)$$

$$N_i(t, \underline{x}) := \mathcal{E}_{,i}^\mu(t, \underline{x}; g) \dot{\mathcal{E}}^\nu(t, \underline{x}; g) g_{\mu\nu}(\mathcal{E}(t, \underline{x}; g)) \quad (1.7)$$

$$-N^2(t, \underline{x}) := \dot{\mathcal{E}}^\mu(t, \underline{x}; g) \dot{\mathcal{E}}^\nu(t, \underline{x}; g) g_{\mu\nu}(\mathcal{E}(t, \underline{x}; g)) - N_i N^i(t, \underline{x}) \quad (1.8)$$

The symplectic form Ω can be written in the equivalent canonical form, with respect to the given foliation as

$$\begin{aligned} \Omega &= \int d^4 X \delta\pi^{\mu\nu} \wedge \delta g_{\mu\nu} = - \int d^4 X \delta\pi_{\mu\nu} \wedge \delta g^{\mu\nu} \\ &= \int d^3 x dt (\delta\tilde{\pi}^{ij} \wedge \delta h_{ij} + \delta\tilde{p} \wedge \delta N + \delta\tilde{p}_i \wedge \delta N^i), \end{aligned} \quad (1.9)$$

where

$$\tilde{\pi}^{ij} := K(t, \underline{x}) (\bar{E}\pi)_{\mu\nu} h^{ik} h^{jl} \mathcal{E}_{,k}^\mu \mathcal{E}_{,l}^\nu \quad (1.10)$$

$$\tilde{p} := -K(t, \underline{x}) \frac{2}{N} (\bar{E}\pi)^{\mu\nu} n_\mu n_\nu \quad (1.11)$$

$$\tilde{p}_i := -K(t, \underline{x}) (\bar{E}\pi)_{\mu\nu} (n^\mu \dot{\mathcal{E}}_i^\nu + n^\nu \dot{\mathcal{E}}_i^\mu) \quad (1.12)$$

Here $K(t, x)$ is the determinant of the transformation from the X to the (t, x) variables,

$$K(t, \underline{x}) = \frac{N(t, \underline{x}) \sqrt{\tilde{h}(t, \underline{x})}}{\sqrt{-g(\mathcal{E}(t, \underline{x}))}}. \quad (1.13)$$

and \tilde{h} is the determinant of the matrix h_{ij} .

The kernel \bar{E} stands for $\bar{E}_{\rho\sigma}^{\mu\nu}(X, X')$, which depends on the chosen foliation; its explicit form is given in [1]. We should note that for a foliation with no metric dependence \bar{E} is the unit operator.

It can be shown that the histories space Π^{cov} is equivalent to the canonical histories space $\Pi^{can} = \times_t(T^*\text{Riem}(\Sigma_t) \times T^*\text{Vec}(\Sigma_t) \times T^*C^\infty(\Sigma_t))$, where $\text{Riem}(\Sigma_t)$ is the space of all Riemannian three-metrics on the surface Σ_t , $\text{Vec}(\Sigma_t)$ is the space of all vector fields on Σ_t , and $C^\infty(\Sigma_t)$ is the space of all smooth scalar functions on Σ_t .

The plan of the paper is as follows. In Section 2. we discuss the status of the symmetries of the theory, in both the covariant and the canonical description. We then discuss the condition for the physical equivalence of canonical quantities related by different choices of foliation, and we write its mathematical expression: the equivariance condition. Furthermore, we elaborate on the relation between the three symmetry groups.

In Section 3. we derive the important result that the histories reduced state space Π^{red} is invariant under spacetime diffeomorphisms. We conclude with some comments on the possible quantisation of gravity within the histories scheme.

2 Invariance Groups

The core of this work is the study of invariance groups of the histories description of general relativity. Starting from the important result of the co-existence of representations of both the diffeomorphism group $\text{Diff}(M)$ and the Dirac algebra of constraints, we first study the way that invariance transformations appear in the covariant and in the canonical description of the histories general relativity. Next, we examine the relations between the invariance groups, and their special role in defining general relativity ‘observables’.

We should mention here that the canonical description of histories general relativity is not an analogue of the standard Lagrangian formulation. We will relate the respective invariance groups of the spacetime *and* the canonical descriptions, and we will construct the spacetime analogue of canonical variables. However, we do not write the Lagrangian action functional, and in this sense we do not directly relate it to its canonical analogue. The direct connection would entail the explicit relation between Lagrangian and Hamiltonian quantities, through a histories analogue of the Legendre transformation.

In what follows, the existence of a representation of the $\text{Diff}(M)$ group will be shown to be of major significance for identifying canonical general

relativity observables.

2.1 Invariance transformations of the covariant description

The dynamical laws of general relativity are invariant under spacetime diffeomorphisms (the group $\text{Diff}(M)$). However, general relativity is characterised by a much larger symmetry group also; these are transformations that are not just point mappings in a given four-dimensional spacetime, but rather diffeomorphisms parameterised by the four-metric g .

2.1.1 $\text{Diff}(M)$ active transformations

The active interpretation of diffeomorphisms transformations highlights one of the main consequences of general covariance: spacetime points have no ontological significance [3]. A related feature is that solutions to the field equations that are related by spacetime diffeomorphisms are regarded as being physically equivalent.

In [4, 1] we have defined the generator of the diffeomorphisms group $\text{Diff}(M)$ to be the generalised ‘Liouville’ function V_W associated with any vector field W on M as

$$V_W := \int d^4X \pi^{\mu\nu}(X) \mathcal{L}_W g_{\mu\nu}(X) \quad (2.1)$$

where \mathcal{L}_W denotes the Lie derivative with respect to W .

These functions V_W , defined for any vector field W , satisfy the Lie algebra of the spacetime diffeomorphism group $\text{Diff}(M)$

$$\{V_{W_1}, V_{W_2}\} = V_{[W_1, W_2]}, \quad (2.2)$$

where $[W_1, W_2]$ is the Lie bracket between vector fields W_1 and W_2 on the manifold M .

The action of V_W on the basic variables of the theory is expressed by infinitesimal diffeomorphisms,

$$\{g_{\mu\nu}(X), V_W\} = \mathcal{L}_W g_{\mu\nu}(X) \quad (2.3)$$

$$\{\pi^{\mu\nu}(X), V_W\} = \mathcal{L}_W \pi^{\mu\nu}(X). \quad (2.4)$$

2.1.2 The histories representation of the Bergmann-Komar group $\mathcal{BK}(M)$

In this section we will show that the covariant formalism of histories canonical general relativity also carries a representation of the group of spacetime mappings that are *functionals* of the four-metric g ; this group was initially introduced by Bergmann and Komar in [2]. We shall start by presenting a brief summary of their construction.

The Bergmann-Komar group $\mathcal{BK}(M)$. Originally motivated by the need to identify the observables in general relativistic theories, Bergmann and Komar studied the relation between the three major invariance groups of the theory, namely the group of spacetime diffeomorphisms, $\text{Diff}(M)$, the group of metric-dependent spacetime diffeomorphisms, $\mathcal{BK}(\mathcal{M})$, and the Dirac algebra of constraints.

The most general type of spacetime transformations are of the form

$$X' = f(X; g) \quad (2.5)$$

which are diffeomorphisms that are functionals of the spacetime metric g . The standard spacetime point mappings in $\text{Diff}(M)$ are a special case of these transformations.

Bergmann and Komar state that, in order to identify general relativity observables, one must find functionals of the field variables that are invariant under these general spacetime mappings. Hence, the three distinct invariance groups provide three distinct criteria for selecting the observables ('gauge-invariant' variables).

An infinitesimal transformation, generated by a vector field ξ , i.e., $\delta x^\mu = x'^\mu - x^\mu = \xi^\mu$, transforms the four-metric $g_{\mu\nu}$, so that

$$\delta g_{\mu\nu} = g'_{\mu\nu} - g_{\mu\nu} = -(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu) = -\mathcal{L}_\xi g_{\mu\nu}. \quad (2.6)$$

In general, the vector field ξ can be an arbitrary functional of the metric, i.e.,

$$\xi^\rho = \xi^\rho(X, g), \quad (2.7)$$

where, for the special case of the normal spacetime diffeomorphisms ξ^ρ is a function of X only:

$$\xi^\rho = \xi^\rho(X). \quad (2.8)$$

Let us now consider the commutator of two consecutive transformations, namely the vector field ξ_c corresponding to $\delta g_{\mu\nu} = \delta_1 \delta_2 g_{\mu\nu} - \delta_2 \delta_1 g_{\mu\nu}$. For the case of (2.8), it is merely the Lie bracket of two vector fields, for example ξ_1 and ξ_2 ,

$$\xi_c^\mu = \xi_{1,\rho}^\mu \xi_2^\rho - \xi_{2,\rho}^\mu \xi_1^\rho = -[\xi_1, \xi_2]^\mu. \quad (2.9)$$

In the more general case (2.7), the commutator is more complicated,

$$\begin{aligned} \xi_c^\mu(X) &= \xi_{1,\rho}^\mu \xi_2^\rho - \xi_{2,\rho}^\mu \xi_1^\rho \\ &= -[\xi_1, \xi_2]^\mu - \int d^4 X' \left\{ \frac{\delta \xi_1^\mu(X)}{\delta g_{\alpha\beta}(X')} \mathcal{L}_{\xi_2 g_{\alpha\beta}(X')} - \frac{\delta \xi_2^\mu(X)}{\delta g_{\alpha\beta}(X')} \mathcal{L}_{\xi_1 g_{\alpha\beta}(X')} \right\}. \end{aligned} \quad (2.10)$$

For the case of $\xi^\rho = \xi^\rho(X)$, we obtain just the expression (2.9). Hence, it is an obvious result that the $\text{Diff}(M)$ group is a subgroup of the enlarged diffeomorphisms group $\mathcal{BK}(M)$.

Next, Bergmann and Komar claim that the Dirac algebra of constraints is a subalgebra of the algebra of $\mathcal{BK}(M)$. In what follows we show the existence of a representation of the $\mathcal{BK}(M)$ on the history space Π^{cov} , and we will study the explicit relation between the two algebras.

The histories space Π^{cov} carries a symplectic representation of the enlarged (metric-dependent) diffeomorphisms group $\mathcal{BK}(M)$. We write the histories generator of the $\mathcal{BK}(M)$ group,

$$U_W := \int d^4 X \pi^{\mu\nu}(X) \mathcal{L}_W g_{\mu\nu}(X), \quad (2.11)$$

where now, the vector field W is a functional of the four-metric g .

We first write the commutators of the generator U_W with the field variables $g_{\mu\nu}$ and $\pi^{\mu\nu}$,

$$\{U_W, g_{\mu\nu}\} = -\mathcal{L}_W g_{\mu\nu} \quad (2.12)$$

$$\{U_W, \pi^{\mu\nu}\} = -\mathcal{L}_W \pi^{\mu\nu} + 2 \int d^4 X' \frac{\delta \xi^\rho(X')}{\delta g_{\mu\nu}(X)} \nabla'_\sigma \pi^{\sigma\tau}(X') g_{\rho\tau}(X'). \quad (2.13)$$

It is straightforward to show that the generators U_W satisfy the Lie algebra of the $\mathcal{BK}(M)$ group,

$$\{U_{W_1}, U_{W_2}\} = U_{W_3}, \quad (2.14)$$

where the vector field W_3 is given by the expression (2.10) with W_3 being the vector field ξ_c^μ of this expression.

In a later section we will examine the relation between the histories representation of the Bergmann-Komar group, the spacetime diffeomorphism group, and the Dirac algebra of constraints.

2.2 Symmetries of the canonical description

In [1] we showed that the Dirac algebra of constraints also appears on the history space. Its generators are³

$$\begin{aligned} \mathcal{H}_\perp(t, \underline{x}) &:= \kappa^2 \tilde{h}^{-1/2}(t, \underline{x}) (\tilde{\pi}^{ij}(t, \underline{x}) \tilde{\pi}_{ij}(t, \underline{x}) - \frac{1}{2} (\tilde{\pi}_i^i)^2(t, \underline{x})) - \\ &\quad \kappa^{-2} \tilde{h}^{1/2}(t, \underline{x}) R(t, \underline{x}) \end{aligned} \quad (2.15)$$

$$\mathcal{H}^i(t, \underline{x}) := -2 \nabla_j \tilde{\pi}^{ij}(t, \underline{x}). \quad (2.16)$$

The smeared form of the super-hamiltonian $\mathcal{H}_\perp(t, \underline{x})$ and the super-momentum $\mathcal{H}_i(t, \underline{x})$ history quantities are defined using as their smearing functions a scalar function L on spacetime M , and a vector field \vec{L} that is spacelike in the sense that

$$L^\mu(X; g) n_\mu(X; g) = 0 \quad (2.17)$$

where the one-form $n_\mu(X; g)$ is the unit normal to the leaves of the foliation. The corresponding covariant expression for the constraints, which is necessary for relating their action to that of the diffeomorphism group, are

$$\mathcal{H}[\vec{L}] = \int d^4 X (\bar{E} \pi)^{\mu\nu} \mathcal{L}_L g_{\mu\nu} + 2 \int d^4 X (\bar{E} \pi)^{\mu\nu} n_\mu n^\rho \mathcal{L}_L g_{\rho\nu} \quad (2.18)$$

$$\mathcal{H}_\perp[L] = \int d^4 X \left[\kappa^2 \frac{N}{\sqrt{-g}} \frac{1}{2} G_{\mu\nu\rho\sigma} (\bar{E} \pi)^{\mu\nu} (\bar{E} \pi)^{\rho\sigma} - \kappa^{-2} \frac{\sqrt{-g}^3}{N} R(h) \right]. \quad (2.19)$$

The tensor $G_{\mu\nu\rho\sigma}$ is the "Dewitt metric"

$$G_{\mu\nu\rho\sigma} = h_{\mu\rho} h_{\nu\sigma} + h_{\mu\sigma} h_{\nu\rho} - h_{\mu\nu} h_{\rho\sigma}, \quad (2.20)$$

³For reasons of simplicity we do not introduce the density $\alpha(t)$ of [1]. In what follows, we use the densities $\tilde{\pi}^{ij}, \tilde{p}_i, \tilde{p}$ instead of the quantities $\pi^{ij} = \alpha(t) \tilde{\pi}^{ij}, p_i = \alpha(t) \tilde{p}_i, p = \alpha(t) \tilde{p}$, which are scalars with respect to time. This means that $\alpha(t)$ does not appear in the expressions for the constraints.

where $h_{\mu\nu} := g_{\mu\nu} + n_\mu n_\nu$.

Furthermore, we add the primary constraints $\Phi(k) = 0$, where

$$\Phi(k) := \int d^4 X (\bar{E}\pi)^{\mu\nu} n_\mu(X; g) k_\nu(X), \quad (2.21)$$

in terms of a smearing one-form k_μ .

It is interesting to note that the supermomentum constraint reads $\mathcal{H}[\vec{L}] = Q[\vec{L}] + 2\Phi(n \cdot \mathcal{L}_L g)$, where

$$Q[\vec{L}] := \int d^4 X (\bar{E}\pi)^{\mu\nu} \mathcal{L}_L g_{\mu\nu}. \quad (2.22)$$

This expression will be used in the calculations that follow.

2.3 A physical requirement for the relation between foliation-dependent variables: the equivariance condition

As already mentioned, the histories approach allows the realisation of a formalism that it is a hybrid of both the covariant and the canonical formalisms.

In [1] we discussed the loss of the spacelike character of a foliation, under a change of the metric. Nevertheless, the introduction of a metric-dependent foliation solved the problem [1].

In addition to this, we need to address the partly independent issue of the $\text{Diff}(M)$ -invariance—or, rather, lack of $\text{Diff}(M)$ -invariance—of the canonical variables. This is a problem even in the standard Lagrangian treatment, because the solution of the initial value problem requires the introduction of a spacetime foliation. In the classical case, the question concerning the dependence of physics on the choice of foliation is easily resolved by specifying a unique Lorentzian metric that solves the classical equations of motion. However, in quantum theory the dependence of the physical results on the choice of foliation is a major issue.

In histories theory we define a representation of the group of spacetime diffeomorphisms. The requirement of the physical equivalence between different choices of time direction for the canonical theory, and the requirement of the spacetime character of the canonical description, are satisfied by means of a simple mathematical condition, namely the *equivariance condition* for the metric-dependent foliations.

To this end, we consider a particular class of metric-dependent embeddings. We denote by $A(\cdot, g]$ any tensor field associated with the embedding, and which is correspondingly a functional of the metric g . The physical requirement is that the change of the tensor field A under a diffeomorphism transformation is compensated by the change due to its functional dependence on g .

Hence, if we consider a diffeomorphism transformation f , and we denote its pull-back operation by f^* , the equivariance condition is given by the expression

$$(f^*A)(\cdot, g] = A(\cdot, f^*g]. \quad (2.23)$$

For an infinitesimal diffeomorphism transformation the equivariance condition is

$$\mathcal{L}_W A(X; g] = \int d^4 X' \frac{\delta A(X; g]}{\delta g_{\mu\nu}(X')} \mathcal{L}_W g_{\mu\nu}(X'). \quad (2.24)$$

In the case of a function $\mathcal{E} : \text{LRiem}(M) \rightarrow \text{Fol}(M)$, (where $\text{LRiem}(M)$ is the space of Lorentzian metrics on M , and $\text{Fol}(M)$ is the space of foliations of M) we say that \mathcal{E} is an ‘equivariant foliation’ if

$$\mathcal{E}(f^*g) = f^{-1} \circ \mathcal{E}(g) \quad (2.25)$$

for all Lorentzian metrics g and $f \in \text{Diff}(M)$.

This concept has the following simple interpretation. Namely, we consider the principle bundle $\text{Diff}(M) \rightarrow \text{LRiem}(M) \rightarrow \text{LRiem}(M)/\text{Diff}(M)$, and then define a left-action ℓ of $\text{Diff}(M)$ on the space of foliations $\text{Fol}(M)$ by

$$\ell_f(\mathcal{F}) := f \circ \mathcal{F} \quad (2.26)$$

for all $f \in \text{Diff}(M)$ and $\mathcal{F} \in \text{Fol}(M)$. We use this action to construct the associated bundle $\text{Fol}(M) \rightarrow \text{LRiem}(M) \times_{\text{Diff}(M)} \text{Fol}(M) \rightarrow \text{LRiem}(M)/\text{Diff}(M)$. Now, in general, if $F \rightarrow P \times_G F \rightarrow X$ is a bundle associated to a principle bundle $G \rightarrow P \rightarrow X$ (via a left action of the group G on F), the cross-sections of the associated bundle are in one-to-one correspondence with functions $\psi : P \rightarrow F$ satisfying the condition

$$\psi(pg) = g^{-1}\psi(p) \quad (2.27)$$

for all $p \in P$ and $g \in G$. It follows, therefore, from Eq. (2.25), Eq. (2.26) and Eq. (2.27), that what we have called an ‘equivariant foliation’ is equivalent to a cross-section of the associated bundle $\text{Fol}(M) \rightarrow \text{LRiem}(M) \times_{\text{Diff}(M)}$

$\text{Fol}(M) \rightarrow \text{LRiem}(M)/\text{Diff}(M)$ over the space $\text{LRiem}(M)/\text{Diff}(M)$ of $\text{Diff}(M)$ -equivalence classes of Lorentzian metrics on M .

As we shall see, the use of equivariant foliations leads to a significant result: the Hamiltonian constraints, the canonical action functional, and the equations of motion on the reduced state space are all invariant under the action of the group of spacetime diffeomorphisms $\text{Diff}(M)$.

2.4 Relation between the invariance groups

We have showed already that in histories theory there exist representations of all three invariance groups of general relativity. We now proceed to study the relations between them.

An immediate observation is that the group of spacetime diffeomorphisms is a subgroup of the Bergmann-Komar group $\mathcal{BK}(M)$. Indeed, the generators of $\text{Diff}(M)$, given by the expression (2.1), are a special case of those of $\mathcal{BK}(M)$ where the vector fields W are *not* functionals of the metric tensor g .

One of the deepest issues to be addressed in canonical gravity is the relation of the algebra of constraints to the spacetime diffeomorphisms group (and, therefore, to the enlarged group $\mathcal{BK}(M)$).

To this end, the commutator of the B-K (Bergmann-Komar) generators with the canonical constraints can be written as

$$\begin{aligned} \{U_W, \Phi(k)\} = & \int d^4X d^4X' \left[\left(\mathcal{L}'_W \bar{E}^{\mu\nu}_{\rho\sigma}(X, X') \right. \right. \\ & - \left. \int d^4X'' \frac{\delta \bar{E}^{\mu\nu}_{\rho\sigma}(X, X')}{\delta g_{\alpha\beta}(X'')} \mathcal{L}_W g_{\alpha\beta}(X'') \right) \pi^{\rho\sigma}(X') n_\mu(X; g) k_\nu(X) \\ & + \left(\bar{E}^{\mu\nu}_{\rho\sigma}(X, X') \pi^{\rho\sigma}(X') k_\nu(X) \int d^4X'' \frac{\delta n_\mu(X)}{\delta g_{\alpha\beta}(X'')} (-\mathcal{L}_W g_{\alpha\beta}(X'')) \right) \\ & \left. + 2 \bar{E}^{\mu\nu}_{\rho\sigma}(X, X') n_\mu(X) k_\nu(X) \int d^4X'' \frac{\delta W^\alpha(X'')}{\delta g_{\rho\sigma}(X')} \nabla''_\tau \pi^{\tau\beta}(X'') g_{\alpha\beta}(X'') \right] \end{aligned} \quad (2.28)$$

$$\begin{aligned} \{U_W, Q(\vec{L})\} = & \int d^4X d^4X' \left[-\frac{\delta \bar{E}^{\mu\nu}_{\rho\sigma}(X, X')}{\delta g_{\alpha\beta}(X'')} \mathcal{L}_W g_{\alpha\beta}(X'') \pi^{\rho\sigma}(X') \mathcal{L}_W g_{\mu\nu}(X) \right. \\ & - \bar{E}^{\mu\nu}_{\rho\sigma}(X, X') \mathcal{L}'_W \pi^{\rho\sigma}(X') \mathcal{L}_W g_{\mu\nu}(X) \mathcal{L}_W g_{\mu\nu}(X) \pi^{\rho\sigma}(X') (\mathcal{L}_{\delta_W L} g_{\mu\nu}(X) - \mathcal{L}_W \mathcal{L}_L g_{\mu\nu}(X) \\ & \left. + 2 \bar{E}^{\mu\nu}_{\rho\sigma}(X, X') \mathcal{L}_W g_{\mu\nu}(X) \int d^4X'' \frac{\delta W^\alpha(X'')}{\delta g_{\rho\sigma}(X')} \nabla''_\tau \pi^{\tau\beta}(X'') g_{\alpha\beta}(X'') \right] \end{aligned} \quad (2.29)$$

$$\begin{aligned}
\{U_W, \mathcal{H}(L)\} &= \int d^4X L(X) \\
&\times \left[\int d^4X' \frac{1}{2} \frac{\delta \tilde{N}}{\delta g_{\alpha\beta}(X')} (-\mathcal{L}_W g_{\alpha\beta})(X') G_{\mu\nu\rho\sigma}(X) (\bar{E}\pi)^{\mu\nu}(X) (\bar{E}\pi)^{\rho\sigma}(X) \right. \\
&+ \frac{\tilde{N}}{2} \frac{\partial G_{\mu\nu\rho\sigma}}{\partial h_{\kappa\lambda}}(X) \left(\delta_{\alpha\beta}^{\kappa\lambda} - \int d^4X' \frac{\delta(n^\kappa n^\lambda)(X)}{\delta g_{\alpha\beta}(X')} \mathcal{L}_W g_{\alpha\beta}(X') \right) (\bar{E}\pi)^{\mu\nu}(X) (\bar{E}\pi)^{\rho\sigma}(X) \\
&- \int d^4X' \tilde{N}(X) G_{\mu\nu\rho\sigma}(X) \int d^4X'' \frac{\delta \bar{E}_{\kappa\lambda}^{\mu\nu}(X, X')}{\delta g_{\alpha\beta}(X'')} \mathcal{L}_W g_{\alpha\beta}(X'') \pi^{\kappa\lambda}(X') (\bar{E}\pi)^{\rho\sigma}(X) \\
&- d^4X' \tilde{N}(X) G_{\mu\nu\rho\sigma}(X) \bar{E}_{\kappa\lambda}^{\mu\nu}(X, X') \mathcal{L}'_W \pi^{\kappa\lambda}(X') (\bar{E}\pi)^{\rho\sigma}(X) \\
&+ \int d^4X' \frac{\delta \tilde{N}^{-1}}{\delta g_{\alpha\beta}(X')} \mathcal{L}_W g_{\alpha\beta}(X')^3 R(X) \\
&- \tilde{N}^{-1}(X)^3 R^{\alpha\beta}(X) \left(\delta_{\kappa\lambda}^{\alpha\beta} - \int d^4X' \frac{\delta(n^{\alpha\lambda} n^\beta)(X)}{\delta g_{\kappa\lambda}(X')} \mathcal{L}_W g_{\kappa\lambda}(X') \right) \\
&+ 2 \int d^4X' \tilde{N}(X) G_{\mu\nu\rho\sigma}(X) \bar{E}_{\kappa\lambda}^{\mu\nu}(X, X') \\
&\times \left(\int d^4X'' \frac{\delta W^\tau(X'')}{\delta g_{\kappa\lambda}(X')} \nabla''_\alpha \pi^{\alpha\beta}(X') g_{\beta\tau}(X'') \right) (\bar{E}\pi)^{\rho\sigma}(X) \Big]. \tag{2.30}
\end{aligned}$$

In these equations we have denoted by $\delta_W L$ the total variation of L under the action of the infinitesimal diffeomorphism generated by the vector field W :

$$\delta_W L^\mu(X) = (\mathcal{L}_W L)^\mu(X) - \int d^4X' \frac{\delta L^\mu(X)}{\delta g_{\alpha\beta}(X')} \mathcal{L}_W g_{\alpha\beta}(X'), \tag{2.31}$$

where $\tilde{N} := N/\sqrt{-g}$.

We note that we have considered the generator $Q(\vec{L})$ rather than the supermomentum constraint, since the latter is a linear combination of $Q(\vec{L})$ with the primary constraint Φ .

The commutators of the $\text{Diff}(M)$ generators with the constraints emerge as a special case of the above equations. In particular,

$$\{V_W, \Phi(k)\} = \Phi(\mathcal{L}_W k) + \int d^4X d^4X' \delta_W [\bar{E}_{\rho\sigma}^{\mu\nu}(X, X') n_\mu(X)] \pi^{\rho\sigma}(X') k_n(X) \tag{2.32}$$

$$\{V_W, Q(\vec{L})\} = Q(\delta_W \vec{L}) + \int d^4X d^4X' \delta_W \bar{E}_{\rho\sigma}^{\mu\nu}(X, X') \mathcal{L}_L g_{\mu\nu} \tag{2.33}$$

$$\begin{aligned}
\{V_W, \mathcal{H}(L)\} &= \mathcal{H}(\mathcal{L}_W L) \\
&+ \int d^4 X d^4 X' d^4 X'' L(X) \pi^{\mu\nu}(X') \pi^{\kappa\lambda}(X'') \delta_W \left(\tilde{N}(X) G_{\mu\nu\rho\sigma}(X) \bar{E}_{\kappa\lambda}^{\mu\nu}(X, X') \bar{E}_{\kappa\lambda}^{\mu\nu}(X, X'') \right) \\
&+ \int d^4 X L(X) \delta_W (\tilde{N}^{-13} R). \tag{2.34}
\end{aligned}$$

We denote by δ_W total variation with respect to the vector field W . Then

$$\delta_W \bar{E}_{\rho\sigma}^{\mu\nu}(X, X') = (\mathcal{L}_W + \mathcal{L}'_W) \bar{E}_{\rho\sigma}^{\mu\nu}(X, X') - \int d^4 X'' \frac{\delta \bar{E}_{\rho\sigma}^{\mu\nu}(X, X')}{\delta g_{\alpha\beta}(X'')} \mathcal{L}_W g_{\alpha\beta}(X''), \tag{2.35}$$

$$\delta_W n_\mu(X) = \mathcal{L}_W n_\mu(X) - \int d^4 X' \frac{\delta n_\mu(X)}{\delta g_{\alpha\beta}(X')} \mathcal{L}_W g_{\alpha\beta}(X'). \tag{2.36}$$

From the above equations we conclude that the action of the diffeomorphism group on the constraints amounts to the action of the diffeomorphisms on the metric-dependent foliation. Hence, the diffeomorphism group generates transformations between the reduced phase spaces that correspond to different metric-dependent foliations.

Next, we impose the equivariance condition on the foliation, which implements the physical principle that histories canonical field variables related by diffeomorphism transformations are physically equivalent. One can show that the terms $\delta_W \bar{E}$ and $\delta_W n$ vanish, and we get

$$\{V_W, \Phi(k)\} = \Phi(\mathcal{L}_W k) \tag{2.37}$$

$$\{V_W, Q(\vec{L})\} = Q(\delta_W \vec{L}) \tag{2.38}$$

$$\{V_W, \mathcal{H}(L)\} = \mathcal{H}(\mathcal{L}_W L). \tag{2.39}$$

Under the infinitesimal symplectic transformation generated by V_W , the constraints transform from $\Phi(k), Q(\vec{L}), \mathcal{H}(L)$ to $\Phi(k'), Q(\vec{L}'), \mathcal{H}(L')$, where

$$k' = k + s \mathcal{L}_W k \tag{2.40}$$

$$\vec{L}' = \vec{L} + s \delta_W \vec{L} \tag{2.41}$$

$$L' = L + s \mathcal{L}_W L \tag{2.42}$$

In particular, if $\vec{L}^\mu n_\mu = 0$, then $(\delta_W \vec{L}^\mu) n_\mu = -\vec{L}^\mu \delta_W n_\mu = 0$, where the vanishing of the last term is due to the equivariance condition for n_μ . This implies that $\vec{L}'^\mu n_\mu = 0$.

From the above, we conclude that *the constraints of canonical general relativity are $\text{Diff}(M)$ -invariant*.

We mentioned in section 2.1.2 that Bergmann and Komar claim that the Dirac algebra is a subalgebra of the Bergmann-Komar algebra. However, this relation was not established in [2] by comparing concrete representations of these algebras. In fact, in our case this relation does not hold, and the generators of the Dirac algebra are *not* special cases of generators of the B-K group. This is because the superhamiltonian is quadratic in momentum, while the B-K group is linear in momentum.

We should also note that the generators of the B-K group do not commute with the constraints, hence the B-K group is not realised on the reduced state space. This leads to the question of whether there exists a different representation of the B-K group which does commute with the constraints (for equivariant foliations) and which has the Dirac algebra as subalgebra. We leave this as an open question.

3 Reduced state space

The study of the parameterised particle system has been proved a helpful model⁴ for developing a histories reduced state space algorithm for general relativity.

We present first, very briefly, the histories reduced state space algorithm, that was originally given in [5].

3.1 Histories treatment of constraints

Classical parameterised systems. Parameterised systems have a vanishing Hamiltonian $H = h(x, p)$, when the constraints are imposed. Classically, two points of the constraint surface C correspond to the same physical state if they are related by a transformation generated by the constraint. The true degrees of freedom correspond to equivalence classes of such points and are represented by points of the reduced state space Γ_{red} .

An element of the reduced state space is *itself* a solution to the classical equations of motion, and it also corresponds to a possible configuration of

⁴The parameterised particle model is considered a good precursor for more complicated parameterised systems, with of course general relativity as the most complicated one.

the physical system at an instant of time; hence the notion of time is unclear, and it is not obvious how to recover the notion of temporal ordering unless we choose an arbitrary gauge-fixing condition.

In the histories approach to parameterised systems, the history constraint surface C_h is defined as the set of all smooth paths from the real line to the constraint surface C . The history Hamiltonian constraint is defined by $H_\kappa = \int dt \kappa(t) h_t$, where $h_t := h(x_t, p_t)$ is first-class constraint. For all values of the smearing function $\kappa(t)$, the history Hamiltonian constraint H_κ generates canonical transformations on the history constraint surface C_h . The history reduced state space Π_{red} is then defined as the set of all smooth paths on the canonical reduced state space Γ_{red} , and it is identical to the space of orbits of H_κ on C_h .

In order for a function on the full state space, Π , to be a physical observable (*i.e.*, to be projectable into a function on Π_{red}), it is necessary and sufficient that it commutes with the constraints on the constraint surface.

Contrary to the canonical treatments of parameterised systems, the classical equations of motion are explicitly realised on the reduced state space Π_{red} . They are given by

$$\{\tilde{S}, F\}(\gamma_{cl}) = \{\tilde{V}, F\}(\gamma_{cl}) = 0 \quad (3.1)$$

where \tilde{S} and \tilde{V} are respectively the action and Liouville functions projected on Π_{red} . Both \tilde{S} and \tilde{V} commute weakly with the Hamiltonian constraint [5]. Furthermore, the equations of motion on Π_{red} remain invariant under time reparameterisations.

Hence, in the histories formalism, parameterised systems have an intrinsic time that does not disappear when we enforce the constraints, either classically or quantum mechanically.

3.2 $\text{Diff}(M)$ invariance of the reduced state space

We showed in section 3.4 that the generators V_W of the spacetime diffeomorphisms group commute with the constraints, and hence they are defined in the reduced state space Π_{red} .

The generator of time translations of the canonical theory is the ‘Liouville’ functional V ,

$$V := \int dt \int d^3 \underline{x} \left\{ \tilde{\pi}^{ij}(t, \underline{x}) \dot{h}_{ij}(t, \underline{x}) + \tilde{p}_i \dot{N}^i + \tilde{p} \dot{N} \right\}. \quad (3.2)$$

It can be easily shown that the Liouville function commutes with the canonical constraints

$$\{V, \Phi(k)\} = 0, \quad (3.3)$$

$$\{V, \mathcal{H}(L)\} = 0, \quad (3.4)$$

$$\{V, \mathcal{H}(\vec{L})\} = 0. \quad (3.5)$$

The Liouville function can be written covariantly as

$$V = \int d^4 X (\bar{E}\pi)_{\rho\sigma} [h^{\mu\rho} h^{\nu\sigma} \mathcal{L}_t g_{\mu\nu} + \mathcal{L}_t(n^\rho n^\sigma)]. \quad (3.6)$$

Here t denotes the deformation vector t^μ associated to our foliation.

In this form, it is easy to show that the Liouville function commutes with the diffeomorphisms:

$$\{V, V_W\} = 0, \quad (3.7)$$

provided that the metric-dependent foliation satisfies the equivariance condition.

The canonical action functional S is defined as

$$S := \int dt \int d^3 \underline{x} \left\{ \tilde{\pi}^{ij}(t, \underline{x}) \dot{h}_{ij}(t, \underline{x}) + \tilde{p}_i \dot{N}^i + \tilde{p} \dot{N} - \mathcal{H}(N) - \mathcal{H}(\vec{N}) \right\}, \quad (3.8)$$

and can clearly be projected onto Π_{red} .

The classical equations of motion *can* be explicitly realised on the reduced state space Π_{red} . They are given by

$$\{\tilde{S}, F\}(\gamma_{cl}) = \{\tilde{V}, F\}(\gamma_{cl}) = 0 \quad (3.9)$$

where \tilde{S} and \tilde{V} are respectively the action and Liouville functions projected on Π_{red} .

The usual dynamical equations for the canonical fields h_{ij} and π^{ij} are equivalent to the history Poisson bracket equations

$$\{S, h_{ij}(t, \underline{x})\}(\gamma_{cl}) = 0 \quad (3.10)$$

$$\{S, \pi^{ij}(t, \underline{x})\}(\gamma_{cl}) = 0 \quad (3.11)$$

where S is defined in Eq. (3.8). The path γ_{cl} is a solution of the classical equations of motion, and therefore corresponds to a spacetime metric that is a solution of the Einstein equations.

Finally, let us note that the canonical action functional S is also diffeomorphic-invariant:

$$\{V_W, S\} = 0. \quad (3.12)$$

The invariance of the action under diffeomorphisms is a rather significant result, and it leads to the conclusion, as one would have anticipated, that the action functional and the equations of motion (3.10–3.11) are the ‘observables’ of general relativity theory, as has been indicated from the Lagrangian treatment of the theory.

We showed that the Liouville function commutes with the Hamiltonian constraints, hence it corresponds to a function \tilde{V} on the reduced state space. In fact, \tilde{V} coincides with the projection of S on Π_{red} .

The key point is that the elements of the reduced state space in the histories formalism are also paths on the standard canonical state space. As such, they preserve the notion of time, in the sense that they are labelled by the external time parameter $t \in \mathbb{R}$.

It is important to remember that the parameter with respect to which the orbits of the constraints are defined, is not in any sense identified with the physical time t . In particular, one can distinguish the paths corresponding to the classical equations of motion by the condition

$$\{F, \tilde{V}\}_{\gamma_{cl}} = 0, \quad (3.13)$$

where F is a functional of the field variables, and γ_{cl} is a solution to the equations of motion.

In standard canonical theory, the elements of the reduced state space are all solutions to the classical equations of motion. In histories canonical theory, however, an element of the reduced state space is a solution to the classical equations of motion only if it also satisfies the condition Eq. (3.13). The reason for this is that the histories reduced state space Π_{red} contains a much larger number of paths (essentially all paths on Γ_{red}). For this reason, histories theory may naturally describe observables that commute with the constraints but which are not solutions to the classical equations of motion.

This last point should be particularly emphasised, because of its possible corresponding quantum analogue. We know that in quantum theory, paths may be realised that *are not* solutions to the equations of motion. My belief

is that the histories formalism will distinguish between instantaneous laws⁵ (namely constraints), and dynamical laws (equations of motion).

Hence, it is possible to have a quantum theory for which the instantaneous laws are satisfied, while the classical dynamical laws are not. This distinction is present, for example, in the history theory of the quantised electromagnetic field, where all physical states satisfy the Gauss law exactly, however electromagnetism field histories are possible which do not satisfy the dynamical equations, i.e., Maxwell's equations. For parameterised systems, this distinction is not possible within the canonical formalism, nevertheless as we explained, it does arise in the histories formalism.

The equations of motion (3.13) imply that physical observables have constant values on the solutions to the classical equations of motion. This need not be the case quantum mechanically, hence quantum realised paths need not be characterised by 'frozen' values of their physical parameters.

4 Some notes on quantisation

It is interesting to examine the histories perspective on quantisation, in the light of these new results. We recall that the canonical quantisation scheme is based upon the canonical commutation relations and a search for their representations on a Hilbert space. However, the canonical commutation relations come originally from considering a *spacetime* metric, with the associated requirement that the induced 3-metric is spacelike; and one facet of the 'problem of time' is to recover a spatio-temporal picture from the purely spatial perspective of the strict canonical formalism.

In the histories approach, this issue can be addressed fully. We seek a representation of the *history commutation relations*, which are defined with reference to the whole of spacetime and not just a 3-surface; in particular, these history variables include a quantised Lorentzian spacetime metric. Finding a representation of the history algebra on a Hilbert space H , makes plausible the possibility of finding a representation the group $\text{Diff}(M)$ on H , and possibly also the Bergmann-Komar group.

⁵A thorough analysis on the connection between instantaneous laws (Gauss' *theorema egregium*) and the dynamical laws of general relativity is presented by Kuchar in [6], where he deduces Einstein's equations starting from the (instantaneous) geometric description of gravity.

In particular, in the histories approach, we can manifestly address a problem that is usually sidestepped in canonical quantisation: namely how to compare the quantum schemes that come from different choices of the ‘internal’ time variable that must be identified to recover a spacetime perspective. This is done by the introduction of the metric-dependent foliations discussed above, which makes explicit the relation between canonical and covariant objects.

The next step is to write the representation of the constraint generators, preferably in a way that the Dirac algebra is preserved. As yet, the history procedure does not provide any preferred strategy for doing this. However, we note that while one may write a quantum history description for any canonical theory [11] (characterised by the presence of a vacuum state)⁶, there exist possible representations of the history algebra, that do *not* have any canonical analogue.

This is particularly relevant to the ‘loop quantum gravity’ approach to quantisation (for its basic features see [12], and also [13] for a recent review). In the canonical treatment, the basic algebra is defined with reference to objects that have support on loops in the three-dimensional surface Σ . In the history version, the relevant objects would be defined on two-dimensional hypersurfaces within the spacetime M .

Alternatively, one could consider path variables corresponding to the $SL(2, \mathbf{C})$ connection on the spacetime M , rather than the $SU(2)$ one of the canonical theory. In that case the representation theory of the history description would be very different from the one of the canonical approach, mainly because the $SL(2, \mathbf{C})$ group is non-compact.

In all cases, the mathematical structures of a quantisation based on histories will conceivably be very different from those in the canonical theory. For this reason, the history construction may uncover substantially different properties from those that arise in the existing approaches to loop quantum gravity. Given that the history approach successfully addresses several key issues that plague the canonical perspective, I believe it provides a promising new approach to tackling the dynamical aspects of the loop quantum gravity programme.

Once we have the constraint operators, we can look for the physical

⁶If H is the Hilbert space of the canonical theory, the corresponding history Hilbert space is a suitable version of the “continuous tensor product” $\otimes_t H_t$

Hilbert space. Technically, this is a problem of the same degree of difficulty as the one that arises in the canonical scheme. However, unless there exist some anomalies in the algebraic relation of the spacetime diffeomorphism group to the constraints, we expect the generators of the diffeomorphism group to exist on the physical Hilbert space, together with the action functional which generates the physical time translations. In this way, the resulting theory will carry both the diffeomorphism symmetry and have a well-defined notion of time-ordering, thus solving another facet of the problem of time in canonical quantum gravity. We shall discuss some simple systems of this type in future work.

However, in order to realise the history quantisation scheme for general relativity, we have to face again the problem of constructing an operator representing the Hamiltonian constraint, which is a formidable task. Or, at least, this would be so if we followed the common wisdom for the quantisation of constrained systems, namely a version of the Dirac approach, upon which our history quantisation algorithm is also based. However, the history theory has more versatility and may provide the concepts and physical predictions of a full quantum theory without needing a Hilbert space structure; for example, by exploiting the geometrical objects on the classical phase space [14]. This provides another potential avenue for quantisation, which may sidestep the problems associated with the intricate construction of a super-Hamiltonian operator.

5 Conclusions

Histories theory in general is characterised by two key ingredients: the history group, and the existence of two distinct generators of time transformations. A significant consequence of this structure is the coexistence of a spacetime and canonical description of a theory. In the case of gravity, this is reflected in the existence of realisations of both the group of spacetime diffeomorphisms, and the Dirac algebra of constraints.

In this paper, we have discussed the relation of these two transformation groups. We focused on the physical equivalence of solutions to the equations of motion associated with different choices of foliation: specifically, if different descriptions are to be equivalent, they should be related by spacetime diffeomorphisms.

We showed how this physical requirement can be satisfied by the introduction of the novel mathematical idea of equivariant foliations. We showed that an immediate consequence is the invariance under spacetime diffeomorphisms of the canonical constraints, the reduced state space action functional, and the equations of motion.

Furthermore, we discussed the enlarged symmetry group of spacetime mappings that are functionals of the four-metric, originally defined by Bergmann and Komar, and we showed that there also exists a representation of this group in the histories theory.

Our results strongly suggest that a quantisation of general relativity based on histories will provide a radically new perspective on the problem of time that has so plagued the canonical approach. In this sense, this paper provides a stepping stone towards the construction of a quantum theory of gravity, that is based on genuine spacetime objects.

Acknowledgements

I would like to especially thank Charis Anastopoulos for very helpful discussions throughout this work. I would like to also thank Chris Isham for his help in the editing of this paper. I gratefully acknowledge support from the EPSRC GR/R36572 grant.

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